OUR LADY OF AFRICA S.S NAMILYANGO (OLAN)

A LEVEL PURE MATHEMATICS SEMINAR QUESTIONS 2024

ORGANISED ON SATURDAY 05TH OCTOBER 2024.

ALGEBRA

1. (a) The sum of n terms of a particular series is given by $S_n = 17n - 3n^2$;
(i) Find an expression for the n^{th} term of the series.
(ii) Show that the series is an Arithmetic progression.

(b) A student deposits shs. 1,200,000 once into her savings account on which an interest of 8% is compounded per annum. After how many years will her balance exceed shs, 200,000?

(c) A piece of land of area $50{,}100m^2$ is divided in such a way that the areas of the plots are in an Arithmetic progression (AP). If the area of the smallest and the largest plots are $2m^2$ and $1000m^2$ respectively, find the;

(i) Number of plots in the piece of land.
(ii) Total area of the first 13 plots to the nearest square metres.

2. (a) Solve the inequality $\frac{x+3}{x-2} \geq \frac{x+1}{x-2}$

(b) Given the curve, $y = \frac{(x-1)(x-4)}{(x-5)}$
(i) Find the range of values of y for which the curve doesnot lie and hence deduce the coordinates of the turning points.
(ii) Show that $y = x$ is an asymptote and state the other asymptote
(iii) Sketch the curve.

3. (a) Solve for x; $64x^{\frac{2}{3}} + x^{\frac{-2}{3}} = 20$
(b) Find the ratio of the coefficient of x^7 to that of x^8 in the expression of $\left(3x + \frac{2}{3}\right)^{17}$
(c)(i) Expand $(1+x)^{-2}$ in descending powers of x including the term in x^{-4}
(ii) If $x = 9$, find the % error in using the first two terms of the expression in c(i) above.

4. (a) Given that W and Z are two complex numbers, solve the simultaneous equations;
$$3Z + W = 9 + 11i$$
$$iW - z = -8 - 2i$$
(b) Use Demoivre's theorem to simplify; $\frac{[\sqrt{3}(cos\theta + isin\theta)]^8}{[3cos2\theta + 3isin2\theta]^3}$
(c) If $(1+3i)z_1 = 5(1+i)$, show that the locus of $|z - z_1| = |z_1|$ where Z is a complex number is a circle and find its Centre and radius
(d) Given that the factors $(x-1)$ *and* $(x+1)$ are factors of the polynomial, $f(x) = ax^4 + 7x^3 + x^2 + bx - 3$, find the values of the constants a and b. Hence, find the set for real values of x for which $f(x) > 0$

TRIGONOMETRY

5. (a) Prove that $\tan(\theta + 60^0)\tan(\theta - 60^0) = \frac{tan^2\theta - 3}{1 - 3tan^2\theta}$

(b) Show that $-\sqrt{5} \leq cosx + 2sinx \leq \sqrt{5}$

(c) Express $10cosxsinx + 12cos2x$ in the form$Rsin(2x + \beta)$, where R is positive and β is an acute angle. Hence find the maximum and minimum values of $10cosxsinx + 12cos2x$ and state clearly the values of x when they occur for $0^0 \leq x \leq 360^0$.

6. (a) Solve the equation: $\frac{4sin^2\theta}{cosec\theta} + \frac{3}{cosec^2\theta sec\theta} = sin^2\theta$ for $0^0 \leq \theta \leq 360^0$

(b) (i) Prove that $\frac{sin2A+cos2A+1}{sin2A+cos2A-1} = \frac{\tan(45^0+A)}{tanA}$

(ii) Show that $\frac{sin\theta cos2\theta+sin3\theta cos6\theta}{sin\theta sin2\theta+sin3\theta sin6\theta} = cot5\theta$

(c) Show that $\frac{sin\theta}{1-cos\theta} = cot\frac{\theta}{2}$. Hence solve $tan\frac{\theta}{2} = \sqrt{3}sin\theta$ for $0^0 \leq \theta \leq 180^0$

7. (a) Given that X, Y, Z are angles of a triangle. Prove that $\tan\left(\frac{X-Y}{2}\right) = \left(\frac{x-y}{x+y}\right) cot\left(\frac{Z}{2}\right)$, hence solve the triangle if $x = 9cm, y = 5.7cm$ and $z = 57^0$

(b) Prove that $Sin[2sin^{-1}(x) + cos^{-1}(x)] = \sqrt{1-x^2}$

(c) Solve the equation; $2\sin(60^0 - x) = \sqrt{2}cos(135^0 + x) + 1$ for $-180^0 \leq x \leq 180^0$

8. (a) If $tanx = \frac{7}{24}$, and $cosy = \frac{-4}{5}$ where x is reflex and y is obtuse, find without using tables or calculators the value of $sin(x + y)$

(b) In a triangle ABC, $\overline{AB} = 10cm, \overline{BC} = 17cm$ and $\overline{AC} = 21cm$ calculate the angle BAC.

(c) Solve the equation $sin3x + sin7x = sin5x$ for $0^0 \leq x \leq 90^0$

(d) (i) Given that $2A + B = 135$ show that $tanB = \frac{tan^2A-2tanA-1}{1-2tanA-tan^2A}$

(ii) If α is an acute angle and $tan\alpha = \frac{4}{3}$, show that $4sin(\theta + \alpha) + 3cos(\theta + \alpha) = 5cos\theta$. Hence solve for θ the equation $4sin(\theta + \alpha) + 3cos(\theta + \alpha) = \frac{\sqrt{300}}{4}$ for $-180^0 \leq \theta \leq 180^0$

ANALYSIS

9. (a) The point (2,1) lies on the curve $Ax^2 + By^2 = 11$ where A and B are constants.

If the gradient of the curve at the point is 6. Find the values of A and B.

(b) One side of a rectangle is three times the other. If the perimeter increases by 2%. What is the percentage increase in the area?

(c) A rectangular box without a lid is made from a thin cardboard. The sides of the base are $2xcm$ and $3xcm$ and the height of the box is hcm. If the total surface area is $200cm^2$, show that $h = \left(\frac{20}{x} - \frac{3x}{5}\right) cm$. And hence find the dimensions of the box to give maximum volume.

10. (a) If $y = \frac{\cos x}{x^2}$, Prove that; $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (2 + x^2)y = 0$

(b) Given the parametric equations $x = 3 + 4\cos\alpha$, $y = 5 - 8\sin\alpha$. Find $\frac{d^2y}{dx^2}$

(c) A curve is defined by the parametric equations $x = t^2 - t$, $y = 3t + 4$. Find the equation of the tangent to the curve at (2,10)

(d) Using calculus of small changes, Show that $\cos 44.6^0 = \frac{\sqrt{2}}{2}\left(\frac{900+2\pi}{900}\right)$

11. (a). Show that $\int_1^{10} x\log x^2 dx = 2\left(50 - \frac{99}{4\ln 10}\right)$

(b) Express $\frac{x^3+9x^2+28x+28}{(x+3)^2}$ into partial fractions, hence or otherwise show that;
$\int_0^1 \frac{x^3+9x^2+28x+28}{(x+3)^2} dx = \frac{1}{3}\left(10 + \ln\frac{4}{3}\right)$

(c) Find the integrals; (i) $\int \ln\left(\frac{2}{x}\right) dx$ (ii) $\int (x\cos x)^2 dx$ (iii) $\int \frac{x}{\sqrt{1-3x}} dx$

12.(a) The pressure in an engine cylinder is given by; $P = 8000[1 - \sin(2\pi t - 3)]\text{Nm}^{-1}$ At what time does this pressure reach a maximum and what is the maximum pressure.

(b) Calculate the area enclosed by the curve $y = \sin x$ and the line $y = \frac{1}{2}$, from $x = 0$ to $x = \pi$ and the x-axis.

(c) The area bounded by the curves $y^2 = 32x$ and $y = x^3$ is rotated about the x-axis through one revolution. Show that the volume of the solid of the solid formed is $\frac{320\pi}{7}$ cubic units

(d) Using Maclaurin's theorem, expand $(x + 1)\sin^{-1}(x)$ up to the term in x^2

13. (a) Using the substitution $y = uv$, solve the differential equation $x^2\frac{dy}{dx} = x^2 + xy + y^2$

(b) Given that $\frac{dy}{dx} = e^{-2y}$ and $y = 0$, when $x = 5$, find the value of x when $y = 3$

(c) Solve the differential equation $(1 + x)\frac{dy}{dx} = xy + xe^x$ given that $y(0) = 1$

(d) The rate at which a liquid runs from a container is proportional to the square root of the depth of the opening below the surface of the liquid. A cylindrical petrol storage tank is sunk in the ground with its axis vertical. There is a leak in the tank at an unknown depth. The level of the petrol in the tank originally full is found to drop by 20cm in 1 hour and by 19cm in the next hour. Find the depth at which the leak is located.

VECTORS

14. (a) Point B is the foot of a perpendicular from point $A\ (3,\ 0, -2)$ to the line $\boldsymbol{r}$ where $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

(i) Find the values of λ corresponding to the point B. hence state the coordinates of B.

(ii) Calculate the distance of the point A from the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and write down the vector parametric equation of the plane containing point A and the line $\boldsymbol{r}$

(b) Find the area of a parallelogram of which the given vectors are adjacent sides, $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{j} + \boldsymbol{k}$ respectively.

(c) A and B are points (3,1,1) and (5,2,3) respectively and C is a point on the line $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. If angle$BAC = 90^0$, find the coordinates of C.

15. (a) Find the coordinates of the point where the line $\frac{x-3}{5} = \frac{3-y}{-2} = \frac{z-4}{3}$ meets the plane $2x - 3y + 7z - 10 = 0$

(b) The vector $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ is perpendicular to the plane containing the line; $\frac{x-3}{-2} = \frac{y+1}{a} = \frac{z-2}{1}$, find the;

(i) Value of a
(ii) Cartesian equation of the plane

(c) Find the perpendicular distance from the point M (4,-3,10) to the line with vector equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

16. (a) Two planes L_1 and L_2 are defined by $3x - 4y + 2z - 5 = 0$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ respectively.

Find;
(i) Cartesian equation of plane L_2
(ii) Acute angle between the two planes
(iii) Vector equation of the line of intersection of L_1 and L_2
(b) Given the points L (2,-1, 0), M (4, 7, 6) and N (8, 5,-4). Find the vector equation of the line which joins the midpoint of LM and MN.
(c) Determine the equation of the plane equidistant from the points A (1, 3, 5) and B (2,-4, 4)

17. (a) Find the equation of the line through point A(1,-2,3) perpendicular to the line $\frac{x-5}{2} = \frac{y-2}{1} = \frac{z-1}{3}$
(b) Prove that Points A (-2,0,6) and B(3,-4,5) lie on opposite sides of the plane $2x - y + 3z = 21$
(c) Find the equation of a plane containing points A (1, 1, 1), B (1, 0, 1) and C (3, 2,-1)
(d) Show that the vectors $2i - j + k, i - 3j - 5k$ and $3i - 4j - 4k$ are coplanar
(e) Point R with position vector **r** divides the line segment AB internally in the ratio $\lambda: \mu$, Show that $r = \frac{a\mu + b\lambda}{\lambda + \mu}$ where a and b are position vectors of A and B respectively. Hence find the position vector of point R which divides AB in the ratio 1:2, given that the position vector of A is $\begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$ and that of B is $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

COORDINATE GEOMETRY

18. (a) A line L passes through the point of intersection of the lines $x - 3y - 4 = 0$ and $y + 3x - 2 = 0$. If L is perpendicular to the line $4y + 3x = 0$, determine the equation of the line L.

(b) Variable point $P(x, y)$ moves such that its distance from point$A(3,0)$ is equal to its distance from the line$x + 3 = 0$. Describe the locus of point P.

(c) Calculate the perpendicular distance between the parallel lines $3x + 4y + 10 = 0$ and $3x + 4y - 15 = 0$

(d) Calculate the area of the triangle which has sides given by the equations $2y - x = 1, y + 2x = 8$ and $4y + 3x = 7$

19. (a) The triangle ABC with vertices A(1,-2), B(7,6) and C(9,2), find:

(i) The equations of the perpendicular bisectors of AB and BC.

(ii) The coordinates of the point of intersection of the perpendicular bisectors

(iii) Find the equation of the circle passing through the three points A,B,C of the triangle above.

(b) Show that the circles $x^2 + y^2 - 2ax + c^2 = 0$ and $x^2 + y^2 - 2by - c^2 = 0$ are orthogonal.

(c) Find the length of the tangent to the circle $x^2 + y^2 - 4x + 9 = 0$ from the point (5,7)

20. (a) Determine the vertex, focus, directrix and axis of the parabola $y^2 - 2y - 8x - 17 = 0$ hence sketch the parabola.

(b) The tangents to the parabola $y^2 = 4ax$ at points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ meet at point T, find the coordinates of T.

(c) If $\left(\frac{1}{2}, 2\right)$ is one extremity of a focal chord of the parabola $y^2 = 8x$, find the coordinates of the other extremity.

(d) If $y = mx + c$ is a tangent to the parabola $y^2 = 4ax$, show that $m = \frac{a}{c}$

21. (a) Show that the parametric equations $x = 1 + 4\cos\theta$ and $y = 2 + 3\sin\theta$ represent an ellipse. Hence determine the coordinates of the centre and the lengths of the semi axes

(b) The normal at the point $P(5\cos\theta, 4\sin\theta)$ on an ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ meets the x and y-axes at A and B respectively. Find the mid-point of the line AB

(c)(i) Find the equation of the tangent to the hyperbola whose points are of the parametric form$\left(2t, \frac{2}{t}\right)$.

(ii) Find the equations of the tangents in (i) which are parallel to $y + 4x = 0$

(iii) Determine the distance between the tangents in c(ii).

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